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Branching of Modular Representations of the Alternating Groups

C. Bessenrodt*

Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 39016 Magdeburg, Germany

and

J. B. Olsson†

Matematisk Institut, Københavns Universitet, 2100 Copenhagen Ø, Denmark

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Based on Kleshchev's branching theorems for the p-modular irreducible representations of the symmetric group and on the recent proof of the Mullineux Conjecture, we investigate in this article the corresponding branching problem for the p-modular irreducible representations of the alternating group A_n. We obtain information on the socle of the restrictions of such A_n-representations to A_{n-1} as well as on the multiplicities of certain composition factors; furthermore, irreducible A_n-representations with irreducible restrictions to A_{n-1} are studied. © 1998 Academic Press

1. INTRODUCTION

The purpose of this paper is to provide information about the restrictions to A_{n-1} of modular irreducible representations of the alternating groups A_n. The results are based on Kleshchev's branching theorems (see

* E-mail: christine.bessenrodt@mathematik.uni-magdeburg.de.
† E-mail: olsson@math.ku.dk.

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[8], [9], [11]) for the symmetric group and on the proof of the Mullineux Conjecture (see [2], [5], [10], [13]).

The starting point in this article is the well-known branching theorem for characteristic 0 representations of the symmetric group S_n. Let S^λ be the irreducible representation of S_n in characteristic 0 labeled by the partition λ of n. Then

$$S^\lambda|_{S_{n-1}} = \bigoplus_A S^{\lambda \setminus A},$$

where the sum is over all removable nodes of λ (see Section 2). On the other hand, the decomposition of the restriction of S^λ to the alternating group A_n is also easy to describe. It depends on the transposition map on partitions, since $S^\lambda \otimes \mathrm{sgn} \simeq S^{\lambda'}$ [6, Sect. 2.5]. Combining these facts, it is quite easy to deduce a branching result for irreducible representations of A_n. This is done in Section 2 to give a perspective of our results in positive characteristic.

Examples show that the branching phenomena in characteristic $p > 0$ are much more complicated. If D^λ is the p-modular irreducible representation of S_n labeled by the p-regular partition λ of n, then $D^\lambda|_{S_{n-1}}$ may also contain composition factors D^μ, where μ is not obtained by removing a node from λ. Moreover, $D^\lambda|_{S_{n-1}}$ is in general not multiplicity-free. The recent progress on this branching problem started with the Jantzen–Seitz conjecture [7] describing the class of p-regular partitions λ for which $D^\lambda|_{S_{n-1}}$ is irreducible. The Jantzen–Seitz conjecture was proved by Kleshchev as a consequence of his much stronger modular branching theorems [8]–[11] (see Theorem 3.1). To exploit Kleshchev's result for investigations of the representations of the alternating groups, it is necessary to understand the restrictions of the modules D^λ to A_n. For $p \neq 2$, these depend on the Mullineux map M on p-regular partitions [13], since

$$D^\lambda \otimes \mathrm{sgn} \simeq D^{\lambda^M}$$

(see [2], [5], [10]). In contrast to the situation at characteristic 0, both the branching and restriction to A_n are quite complicated in characteristic p, and to deal with the representations of the alternating groups, we have to control both processes simultaneously. This is possible using the residue symbols and signature sequences introduced by the authors in [2]. In the case of characteristic 2 we have to invoke a result of Benson [1].

The paper is organized as follows. In Section 2 we treat branching for representations of A_n in characteristic 0. In Section 3 we present the necessary background dealing with positive characteristic, that is, Kleshchev's results and residue symbols. In the next section, we describe the classification of the modular irreducible A_n-representations based on

Benson's result at $p = 2$ and on the Mullineux map at $p \neq 2$. Section 5 deals with the case of odd characteristic, and Section 6 with the case of characteristic 2. We obtain information on the socle of the restrictions of irreducible A_n-representations to A_{n-1} as well as on the multiplicities of certain composition factors of these restrictions. Furthermore, we describe the labels of those irreducible modules of A_n that remain irreducible upon restriction to A_{n-1} at characteristic $p \neq 2$, and combinatorial conditions for such labels are given at $p = 2$.

For general facts on representations of the symmetric groups, we refer the reader to [6]. Throughout the paper we will always assume that our representations are over fields that are splitting fields for the alternating groups, e.g., one may take the fields to be algebraically closed.

2. BRANCHING OF REPRESENTATIONS OF THE ALTERNATING GROUPS AT CHARACTERISTIC 0 OR LARGE p

Let λ be a partition of n. It is well known that at characteristic 0 resp., at prime characteristic p with $p > n$, the character $[\lambda]$ resp., the corresponding representation of S_n splits on restriction to A_n if and only if λ is symmetric, i.e., $\lambda = \lambda'$. In this case, it splits into two (via the transposition (12)) conjugate A_n-characters $\{\lambda\}^+$ and $\{\lambda\}^-$. So for a branching formula for A_n-characters, one needs to study conjugation properties together with the usual well-known branching of ordinary irreducible S_n-representations.

First we recall some notations and definitions for partitions. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n, i.e., $\lambda_1, \dots, \lambda_k$ are integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_{i=1}^{k} \lambda_i = n$. Then

$$Y(\lambda) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | 1 \leq i \leq k, 1 \leq j \leq \lambda_i\} \subset \mathbb{Z} \times \mathbb{Z}$$

is the *Young diagram* of λ, and $(i,j) \in Y(\lambda)$ is called a *node* of λ. If $A = (i,j)$ is a node of λ and $Y(\lambda) \setminus \{(i,j)\}$ is again a Young diagram of a partition, then A is called a *removable* node, and $\lambda \setminus A$ denotes the corresponding partition of $n - 1$.

Similarly, if $A = (i,j) \in \mathbb{N} \times \mathbb{N}$ is such that $Y(\lambda) \cup \{(i,j)\}$ is the Young diagram of a partition of $n + 1$, then A is called an *indent* node of λ.

The Nonsymmetric Case: $\lambda \neq \lambda'$

Assume that λ has a removable node A such that

$$\lambda \setminus A = (\lambda \setminus A)' = \lambda' \setminus B$$

for a suitable node B in λ'. In this case we say that λ is *almost symmetric*.

Since λ is not symmetric, the "conjugate node" A' to A does not belong to λ. Hence for every other removable node $C \neq A$ in λ we have

$$\lambda \setminus C \neq (\lambda \setminus C)'.$$

Even stronger: for every pair of removable nodes $C \neq A$, $\tilde{C}$ in λ, we have

$$\lambda \setminus C \neq (\lambda \setminus \tilde{C})'.$$

Now, if we assume that there is no node A as above, then we can easily see that also in this case for every pair of removable nodes C, $\tilde{C}$ in λ, we have

$$\lambda \setminus C \neq (\lambda \setminus \tilde{C})'.$$

Thus we obtain for the branching of the character $\{\lambda\}$:

(i) If λ is almost symmetric (with respect to the node A), then

$$\{\lambda\}|_{A_{n-1}} = \widehat{\{\lambda \setminus A\}} + \sum_{C \neq A} \{\lambda \setminus C\},$$

where $\widehat{\{\mu\}} = \{\mu\}^+ + \{\mu\}^-$ for a symmetric partition μ, and the partitions $\lambda \setminus C$, $C \neq A$, are all pairwise nonconjugate, so that the restriction is multiplicity-free.

(ii) If λ is not almost symmetric, then

$$\{\lambda\}|_{A_{n-1}} = \sum_{C} \{\lambda \setminus C\},$$

where the partitions $\lambda \setminus C$ are all pairwise nonconjugate, so that again the restriction is multiplicity-free.

The Symmetric Case: $\lambda = \lambda'$

In this case there exists at most one removable node A (on the main diagonal) with

$$\lambda \setminus A = (\lambda \setminus A)'.$$

Moreover, it is clear that for every other removable node $C \neq A$ in λ, we have a removable node $\tilde{C} \neq C$ in λ with

$$\lambda \setminus C = (\lambda \setminus \tilde{C})'.$$

Before we can state the restrictions of the two conjugate characters $\{\lambda\}^+$ and $\{\lambda\}^-$ to A_{n-1}, we first have to be precise about the choice of our labeling (see [6, Sect. 2.5]). The only critical conjugacy class of A_n where the characters $\{\lambda\}^+$ and $\{\lambda\}^-$ differ are the two classes of cycle type $h(\lambda) = (h_{11}, \ldots, h_{rr})$, where h_{ii} denotes the length of the ith principal hook of λ, and r is the order of the Durfee square of λ. Let

$$\sigma_{h(\lambda)^+} = (1 \cdots h_{11})(h_{11} + 1 \cdots h_{11} + h_{22}) \cdots \left(\sum_{i=1}^{r-1} h_{ii} + 1 \cdots \sum_{i=1}^{r} h_{ii} \right)$$

be an element in the conjugacy class of cycle type $h(\lambda)$ and of $+$-type, and choose the labeling of the characters such that

$$\{\lambda\}^{\pm}(\sigma_{h(\lambda)^+}) = \tfrac{1}{2}\left([\lambda](\sigma_{h(\lambda)}) \pm \sqrt{[\lambda](\sigma_{h(\lambda)}) \prod_{i=1}^{r} h_{ii}} \right).$$

If $\lambda = \lambda'$ has a removable node A on the main diagonal, then $h_{rr} = 1$, and so $h(\lambda \setminus A) = (h_{11}, \ldots, h_{r-1\, r-1})$. The sign choice for the conjugacy classes in A_{n-1} of type $(h_{11}, \ldots, h_{r-1\, r-1})$ is made compatibly with the choice above, and the signs for the characters are chosen as before. Then

$$\{\lambda \setminus A\}^{\pm}(\sigma_{h(\lambda \setminus A)^+}) = \tfrac{1}{2}\left([\lambda \setminus A](\sigma_{h(\lambda \setminus A)}) \pm \sqrt{[\lambda](\sigma_{h(\lambda)}) \prod_{i=1}^{r-1} h_{ii}} \right).$$

Hence we have the following branching behavior: (i) If λ has a removable node A on the main diagonal, then

$$\{\lambda\}^{\pm}|_{A_{n-1}} = \{\lambda \setminus A\}^{\pm} + \sum_{C \neq A} \{\lambda \setminus C\},$$

where C only runs through the removable nodes of λ above the main diagonal.

(ii) If λ has no removable node on the main diagonal, then

$$\{\lambda\}^{\pm}|_{A_{n-1}} = \sum_{C} \{\lambda \setminus C\},$$

where C only runs through the removable nodes of λ above the main diagonal.

In particular, the restrictions are again multiplicity-free.

In the discussion above, some of the properties become even more obvious when one uses the Frobenius symbol displaying the principal hooks of λ rather than working in terms of the parts of λ.

3. MODULAR BRANCHING OF S_n-REPRESENTATIONS AND RESIDUE SYMBOLS

In this section we collect some definitions and results that are needed in the following sections.

Let p be a prime; this is needed in the representation theoretic context, for the combinatorial statements p may be an arbitrary odd integer > 1.

We define a (p)-*signature sequence* X as a sequence

$$X: c_1\varepsilon_1\ c_2\varepsilon_2\ \cdots\ c_s\varepsilon_s,$$

where the c_i are residues mod p, and the ε_i are signs, for $i = 1, \dots, s$. For $0 \le i \le s$ and $0 \le \alpha \le p-1$ we define

$$\sigma_\alpha^X(i) = \sum_{\{k \le i | c_k = \alpha\}} \varepsilon_k;$$

here we make the conventions that an empty sum is 0 and that $+$ is counted as $+1$ and $-$ as -1 in the sum. The *end value* σ_α^X of α in X is then defined to be

$$\sigma_\alpha^X = \sigma_\alpha^X(s).$$

Furthermore, we define the *peak value* of α in X to be

$$\pi_\alpha(X) = \max\{\sigma_\alpha^X(i) | 0 \le i \le s\}.$$

We call $c_i \in X$ *normal* of residue α if $\sigma_\alpha^X(i) > \sigma_\alpha^X(j)$ for all $j \le i-1$ and $\sigma_\alpha^X(i) > 0$. This is only possible when $c_i\varepsilon_i = \alpha+$. In this case we also call $\sigma_\alpha^X(i)$ the *height* $\operatorname{ht} c_i$ of c_i. Moreover, $c_i \in X$ is called *good* of residue α (for short also α-good) if c_i is normal of residue α and i is minimal with

$$\sigma_\alpha^X(i) = \pi_\alpha(X).$$

Note that if c_i is good of residue α, then $\operatorname{ht} c_i = \pi_\alpha(X)$.

Before we associate these definitions with partitions, we recall some further notation and definitions for partitions.

A partition $\lambda = (l_1^{a_1}, l_2^{a_2}, \dots, l_t^{a_t})$ is called *p-regular* if $1 \le a_i \le p-1$ for $i = 1, \dots, t$. The p-regular partitions of n label in a canonical way the p-modular irreducible representations D^λ of S_n [6].

The *p-residue* of a node $A = (i, j)$ in the Young diagram of λ is defined to be the residue modulo p of $j - i$, denoted res $A = j - i \pmod p$. The *p-residue diagram* of λ is obtained by writing the p-residue of each node of the Young diagram of λ in the corresponding place.

EXAMPLE. Take $p = 5$, $\lambda = (6^2, 5, 4)$. Then the 5-residue diagram of λ is

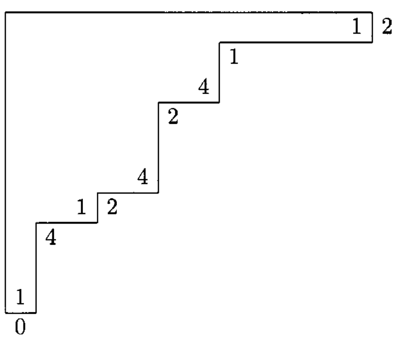
$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 & \\ 2 & 3 & 4 & 0 & & \end{array}$$

For a partition λ, its *node sequence* $N(\lambda)$ is a signature sequence defined as follows. Let $A_1, \dots, A_l$ be all of the indent and removable nodes of λ, read successively from top to bottom and from right to left in the Young diagram of λ. For each $i \in \{1, \dots, r\}$, let c_i be the residue of A_i and let the sign ε_i be $+$ if A_i is removable and $-$ if A_i is an indent node. Then the node sequence is defined to be

$$N(\lambda)\colon c_1\varepsilon_1 \; c_2\varepsilon_2 \; \cdots \; c_s\varepsilon_s.$$

A removable node A of λ is then called *normal* (resp., *good*) if the corresponding entry in the node sequence is normal (resp., good); the *height* ht A is then the height of this entry. The *α-peak value* of λ is the peak value of α in its node sequence.

EXAMPLE. Let $p = 5$ and $\lambda = (12, 7^2, 5^3, 3, 1^3)$. Below, we have only indicated the removable and indent nodes in the 5-residue diagram of λ.



Then the node sequence of λ is

$$N(\lambda)\colon 2- \quad \underline{1+} \quad 1- \quad \underline{4+} \quad 2- \quad \underline{4+} \quad 2- \quad 1+ \quad 4- \quad \underline{1+} \quad 0-,$$

where we have underlined the normal entries in the node sequence. Note that then the corresponding removable nodes of λ are also normal.

These combinatorial concepts play a vital rôle in the following theorem due to Kleshchev (see [8]–[11]).

THEOREM 3.1. *Let* λ *be a p-regular partition of* $n, n \in \mathbb{N}, n \geq 2$. *Then the following holds*:

(i) $soc(D^{\lambda}|_{S_{n-1}}) \simeq \bigoplus_{A \text{ good}} D^{\lambda \setminus A}$.

(ii) $D^{\lambda}|_{S_{n-1}}$ *is completely reducible if and only if all normal nodes in* λ *are good*.

(iii) Let A be a removable node of λ such that $\lambda \setminus A$ is p-regular. Then the multiplicity of $D^{\lambda \setminus A}$ in $D^{\lambda}|_{S_{n-1}}$ is given by

$$\left[D^{\lambda}|_{S_{n-1}} : D^{\lambda \setminus A} \right] = \begin{cases} \text{ht } A & \text{if } A \text{ is normal in } \lambda, \\ 0 & \text{else}. \end{cases}$$

As a consequence of this theorem one can determine the p-regular partitions λ of n for which the restriction $D^{\lambda}|_{S_{n-1}}$ is irreducible. We recall the following definition:

DEFINITION 3.2. Let $\lambda = (l_1^{a_1}, \ldots, l_t^{a_t})$ be a p-regular partition, where $l_1 > l_2 > \cdots > l_t$, $0 < a_i < p$ for $i = 1, \ldots, t$. Then λ is called a *JS-partition* if its parts satisfy the congruence

$$l_i - l_{i+1} + a_i + a_{i+1} \equiv 0 \bmod p \qquad \text{for } 1 \leq i < t.$$

The type α of λ is defined by $\alpha \equiv l_1 - a_1 \bmod p$.

Note that the type of a JS-partition is just the residue of the unique normal (and thus good) node at the top corner of λ. The letters JS are an abbreviation of Jantzen and Seitz; these authors conjectured the equivalence of (i) $\Leftrightarrow$ (iii) in the following corollary (see [3], [8], [9]). We denote the partition obtained by removing the ith corner node from λ by $\lambda(i)$.

COROLLARY 3.3. *With notation as above, the following are equivalent*:

(i) $D^{\lambda}|_{S_{n-1}}$ *is irreducible* (*and in this case*, $D^{\lambda}|_{S_{n-1}} \simeq D^{\lambda(1)}$).

(ii) λ *has exactly one normal node* (*which is then the only good node in* λ).

(iii) λ *is a JS-partition*.

For later purposes we also have to introduce a different notation for p-regular partitions, which will allow us to have both the removal of good nodes and the p-analogue of conjugation given by the Mullineux map (defined in the next section) under control.

Let λ be a p-regular partition of n. The *p-rim* of λ is a part of the rim of λ [6, p. 56], which is composed of *p-segments*. Each p-segment, except possibly the last, contains p points. The first p-segment consists of the first p points of the rim of λ, starting with the longest row. (If the rim contains at most p points, it is the entire rim.) The next segment is obtained by starting in the row next below the previous p-segment. This process is continued until the final row is reached. We let a_1 be the number of nodes in the p-rim of λ and let r_1 be the number of rows in λ; if a_1 is a multiple of p, the p-rim is called *p-singular*. Removing the p-rim of λ, we get a p-regular partition of $n - a_1$, and we let a_2, r_2 be the length of the p-rim and the number of parts of this new partition, respectively. Continuing in this way, we get the sequences of numbers $a_1, \ldots, a_m, r_1, \ldots, r_m$.

The *residue symbol* $R_p(\lambda)$ of λ is then defined as

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix},$$

where x_j is the residue of $a_{m+1-j} - r_{m+1-j}$ modulo p, and y_j is the residue of $1 - r_{m+1-j}$ modulo p. The partition λ can be recovered from the residue symbol $R_p(\lambda)$ as the unique p-regular partition with this residue symbol (see [2]).

A column $\substack{x_i \\ y_i}$ in the residue symbol is called *singular* if $y_i = x_i + 1$ and *regular* otherwise. Note that a residue symbol can never start with the singular column $\substack{0 \\ 1}$, as this does not correspond to a p-regular partition.

Now we define the *Mullineux* (signature) *sequence* $M(\lambda)$ via the residue symbol of λ. With $R_p(\lambda)$ as above, we set

$$\begin{aligned} M(\lambda) = 0- \quad & x_1+ \quad (x_1+1)- \quad y_1+ \quad (y_1-1)- \\ & x_2+ \quad (x_2+1)- \quad y_2+ \quad (y_2-1)- \\ & \qquad\quad \vdots \qquad\qquad\quad \vdots \\ & x_m+ \quad (x_m+1)- \quad y_m+ \quad (y_m-1)- \end{aligned}.$$

Starting with the signature $0-$ corresponds to starting with an empty partition at the beginning, which just has the indent node $(1,1)$ of residue 0.

In [2] the following result was proved.

THEOREM 3.4. *Let λ be a p-regular partition. Then for all $\alpha, 0 \le \alpha \le p-1$ we have*

$$\pi_\alpha(M(\lambda)) = \pi_\alpha(N(\lambda)).$$

As a consequence, we can recognize the normal and good nodes of λ also in $R_p(\lambda)$:

COROLLARY 3.5. *The following statements are equivalent for a p-regular partition λ $(0 \le \alpha \le p-1)$:*

(i) *λ has a normal (good) node of residue α.*

(ii) *$M(\lambda)$ has a normal (good) entry of residue α.*

In [2] the following result on the behavior of residue symbols with respect to removing good nodes was proved:

THEOREM 3.6. *Suppose that the p-regular partition λ has a good node A of residue α. Let*

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix}.$$

Then for some j, $1 \le j \le m$, one of the following occurs:

(1) x_j *is α-good for $M(\lambda)$ and*

$$R_p(\lambda \setminus A) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_j - 1 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_j & \cdots & y_m \end{Bmatrix}.$$

(2) y_j *is α-good for $M(\lambda)$, and in this case if $j = 1$ and $\alpha = 0$, then $x_1 = 0$ and*

$$R_p(\lambda \setminus A) = \begin{Bmatrix} x_2 & \cdots & x_m \\ y_2 & \cdots & y_m \end{Bmatrix},$$

i.e., the first column is removed, or else if $(j, \alpha) \neq (1, 0)$, then

$$R_p(\lambda \setminus A) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_j & \cdots & x_m \\ y_1 & y_2 & \cdots & y_j + 1 & \cdots & y_m \end{Bmatrix}.$$

4. MODULAR IRREDUCIBLE A_n-REPRESENTATIONS

For a classification of the modular irreducible A_n-representations, we need to know which modular irreducible S_n-representations split when restricted to A_n. For $p = 2$, the answer was given by Benson [1]:

THEOREM 4.1. *Let $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ be a 2-regular partition of $n \in \mathbb{N}$, $n \ge 2$, D^λ the corresponding modular irreducible S_n-representation. Then the restriction $D^\lambda|_{A_n}$ is reducible if and only if the parts of λ satisfy the following two conditions* (*where we set $\lambda_{m+1} = 0$ if m is odd*):

(i) $\lambda_{2j-1} - \lambda_{2j} \le 2$ *for all j.*

(ii) $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod 4$ *for all j.*

If $D^\lambda|_{A_n}$ is reducible, then it splits into two nonisomorphic irreducible summands, say $D^\lambda|_{A_n} \simeq C^{\lambda^+} \oplus C^{\lambda^-}$.

We call the 2-regular partitions satisfying the conditions (i) and (ii) above *S-partitions*.

From Theorem 4.1 we can now deduce the classification of the 2-modular irreducible A_n-representations:

COROLLARY 4.2. *A complete list of* 2*-modular irreducible* A_n*-representations is given by*

$$\begin{array}{ll} C^{\lambda^+}, C^{\lambda^-} & \textit{for } \lambda \vdash n \textit{ a } 2\textit{-regular } S\textit{-partition}, \\ C^{\lambda} & \textit{for } \lambda \vdash n \textit{ a } 2\textit{-regular non-}S\textit{-partition}. \end{array}$$

For $p \neq 2$, the split restrictions are those of modules fixed by tensoring with the sign representation.

This case is determined by the Mullineux map, which we describe on the residue symbols:

DEFINITION 4.3. Let the residue symbol of the p-regular partition λ be

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{Bmatrix}.$$

Then the Mullineux conjugate λ^M is the p-regular partition whose residue symbol is

$$R_p(\lambda^M) = \begin{Bmatrix} \varepsilon_1 - y_1 & \cdots & \varepsilon_m - y_m \\ \varepsilon_1 - x_1 & \cdots & \varepsilon_m - x_m \end{Bmatrix},$$

where

$$\varepsilon_j = \begin{cases} 1 & \text{if } x_j + 1 = y_j, \\ 0 & \text{otherwise}. \end{cases}$$

As a consequence of Theorem 3.6, we state the following property of good nodes with respect to the Mullineux map:

THEOREM 4.4 [2], [5]. *Let* λ *be a* p*-regular partition, and let* A *be a good node of* λ*. Then there exists a good node* B *of the Mullineux image* λ^M *such that* $(\lambda \setminus A)^M = \lambda^M \setminus B$.

In 1979, Mullineux conjectured that the map M defined above describes the result of tensoring an irreducible representation by the sign representation. After a long struggle this conjecture was finally solved by the work of Kleshchev and Ford and Kleshchev. As an application of his Branching Theorems for modular S_n-representations, Kleshchev had reduced the Mullineux Conjecture to the purely combinatorial statement in Theorem 4.4 on the removal of good nodes from a partition, which was then proved by Ford and Kleshchev.

THEOREM 4.5. *Let λ be a p-regular partition. Then*

$$D^{\lambda} \otimes \operatorname{sgn} \simeq D^{\lambda^M}.$$

Based on this and Clifford theory, we have a combinatorial description for the splitting of the modular irreducible S_n-representations over A_n also for odd primes p (see [4]):

THEOREM 4.6. *Let p be an odd prime, and let $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ be a p-regular partition of n, D^{λ} the corresponding modular irreducible S_n-representation. Then the restriction $D^{\lambda}|_{A_n}$ is reducible if and only if λ is a fixed point under the Mullineux map, i.e., $\lambda^M = \lambda$. If $D^{\lambda}|_{A_n}$ is reducible, then it splits into two nonisomorphic irreducible summands C^{λ^+}, C^{λ^-}, so $D^{\lambda}|_{A_n} \simeq C^{\lambda^+} \oplus C^{\lambda^-}$.*

As before, this implies the classification of the p-modular irreducible A_n-representations [4]:

COROLLARY 4.7. *A complete list of p-modular irreducible A_n-representations is given by the modules*:

$$\begin{array}{ll} C^{\lambda^+}, C^{\lambda^-} & \lambda \vdash n \text{ } p\text{-regular}, \ \lambda = \lambda^M \\ C^{\lambda} = C^{\lambda^M} & \lambda \vdash n \text{ } p\text{-regular}, \ \lambda \neq \lambda^M. \end{array}$$

5. BRANCHING AND p-CONJUGATION AT ODD p

In this section, we always assume that $p > 2$; otherwise p is an arbitrary odd integer in the combinatorial statements and has to be a prime in the representation theoretic statements. Recall the phenomena occurring "at characteristic 0" (described in Section 2) to imagine what to expect now. In the following, we denote by F a field of characteristic p, again assumed to be sufficiently large. First a lemma:

LEMMA 5.1. *Let M be an FS_n-module. Then*

$$\operatorname{soc}(M|_{A_n}) = \operatorname{soc}(M)|_{A_n}.$$

Proof. Since A_n is normal in S_n and of index prime to p, the Jacobson radicals of FS_n and FA_n are related by $J(FS_n) = J(FA_n)FS_n$, due to a result by Villamayor [12, p. 524]. Since the socle of a module is the set of elements annihilated by the Jacobson radical [15, Theorem 1.8.18], the assertion follows. ∎

For convenience, we let C^{λ^o} denote the module $C^{\lambda^o} = C^{\lambda}$ if λ is not a Mullineux fixed point, and $C^{\lambda^o} \in \{C^{\lambda^+}, C^{\lambda^-}\}$ if λ is a Mullineux fixed point.

PROPOSITION 5.2. *$D^{\lambda}|_{S_{n-1}}$ is completely reducible if and only if $C^{\lambda^o}|_{A_{n-1}}$ is completely reducible.*

Proof. This follows immediately from the preceding Lemma and Theorem 4.6. ∎

In the proof of the following result we require the content vector of a partition λ, which is defined to be $c(\lambda) = (c_0, \dots, c_{p-1})$, where c_i is the number of nodes in the p-residue diagram of λ, which are of residue i. The basic fact we need is that with $c(\lambda)$ as above, the content vector of the Mullineux conjugate is $c(\lambda^M) = (d_0, \dots, d_{p-1})$, where $d_\alpha = c_{-\alpha}$ (see [2], [14]).

THEOREM 5.3 (The almost-p-symmetric case). *Let λ be p-regular, and assume $\lambda \neq \lambda^M$. Furthermore, assume that λ has a good node A with $\lambda \setminus A = (\lambda \setminus A)^M$.*

(i) *Then all of the columns in the residue symbol of λ are Mullineux-fixed, except for one column,*

$$\begin{matrix} x \\ y \end{matrix} = \begin{cases} \begin{matrix} \boxed{x} \\ 1 - x \end{matrix} & \text{with } x \neq 0, \\ \text{or} & \\ \begin{matrix} x \\ \boxed{-1 - x} \end{matrix} & \text{with } x \neq -1, \end{cases}$$

where the marked entry corresponds to the good node A in the sense of Theorem 3.6. In particular, res $A \neq 0$.

(ii) *There are no normal nodes B and $\tilde{B}$ with $B \neq A \neq \tilde{B}$ such that $\lambda \setminus B$ and $\lambda \setminus \tilde{B}$ are p-regular and satisfy*

$$\lambda \setminus B = \left(\lambda \setminus \tilde{B}\right)^M.$$

In particular, there are no good nodes B and $\tilde{B}$ different from A for which this holds.

Proof. (i) Theorem 3.6 immediately tells us that all of the columns in the residue symbol of λ are Mullineux-fixed, except for the one column with the good entry corresponding to the good node A. As λ itself is not Mullineux-fixed, this exceptional column cannot be a 0-column at the start of the residue symbol, since this would vanish on removing A.

So let us assume that $\begin{smallmatrix} x \\ y \end{smallmatrix}$ is the exceptional column, and consider first the case where the upper entry x is the good entry corresponding to A. Then the corresponding column $\begin{smallmatrix} x-1 \\ y \end{smallmatrix}$ in $R_p(\lambda \setminus A)$ has to be Mullineux-fixed. If this column is regular, then we have $y = 1 - x$ and $x \neq 0$, as claimed. If we assume it is singular, then $y = x$, contradicting the assumption that the upper entry x in $R_p(\lambda)$ was good. Similarly, if the lower entry y in the exceptional column is the good entry corresponding to A, then in the case of the corresponding column $\begin{smallmatrix} x \\ y+1 \end{smallmatrix}$ for $\lambda \setminus A$ being regular, we obtain $y = -x - 1$, and we must have $x \neq -1$, since otherwise in the (singular) column $\begin{smallmatrix} x \\ y \end{smallmatrix} = \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}$ the lower entry is not good. If we assume $\begin{smallmatrix} x \\ y+1 \end{smallmatrix}$ to be singular, then $y = x$ and the fixed point condition gives $x = -x$; hence in this case the exceptional column is $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ and hence is Mullineux-fixed, contradicting our assumption.

The conditions we have derived above immediately imply that res $A \neq 0$.

(ii) Let $c(\lambda) = (c_0, c_1, \ldots, c_{p-1})$ be the content vector of λ, and let $r =$ res A, so $r \neq 0$ by (i). Then the condition $\lambda \setminus A = (\lambda \setminus A)^M$ implies $c_r - 1 = c_{p-r}$ and $c_i = c_{p-i}$ for all residues $i \neq r, p - r$. Now let B, $\tilde{B}$ be nodes of λ, different from A, with residues i and j, respectively, such that their removal gives p-regular partitions with $\lambda \setminus B = (\lambda \setminus \tilde{B})^M$. In the first part of the proof we show that $i = r = j$; here we do not need the condition that these nodes are normal (and that A is good). Since the Mullineux map leaves the content of residue 0 invariant, either none or both of B and $\tilde{B}$ are of residue 0. If $i = 0 = j$, then $c_k = c_{p-k}$ for all residues $k \neq 0$, a contradiction, since $r \neq 0$. Hence $i \neq 0 \neq j$.

We assume first that $i \neq r \neq j$. If $j \neq p - i$, then comparing the content at residue i and using the relation above gives

$$c_i - 1 = c_{p-i} = c_i,$$

a contradiction. If $j = p - i$, then we obtain $c_k = c_{p-k}$ for all residues k, again a contradiction.

Hence we only have to consider the case where $B \neq A \neq \tilde{B}$ are normal nodes of λ with residues $i = r = j$. As these nodes are normal but not good, the r-peak value goes down by 2 on removing these nodes (the $+1$-contribution before the good node is replaced by a -1-contribution); hence

$$\begin{aligned} \pi_r\big(\lambda \setminus \tilde{B}\big) &= \pi_r(\lambda) - 2 \\ &= \pi_r\big((\lambda \setminus B)^M\big) \\ &= \pi_{-r}(\lambda \setminus B) \\ &= \pi_{-r}(\lambda) + \varepsilon(B), \end{aligned}$$

where $\varepsilon(B) \in \{0,1\}$; the last equation follows as res $B = r \neq -r$ $(r \neq 0)$, and hence the effect of removing B can at most be an increase of 1 in the peak values for the residues $r \pm 1$.

On the other hand, removing a good node only decreases the peak value for the corresponding residue by 1, and the effect on other residues is similar to that above, so

$$\begin{aligned}\pi_r(\lambda) - 1 = \pi_r(\lambda \setminus A) &= \pi_{-r}\big((\lambda \setminus A)^M\big) \\ &= \pi_{-r}(\lambda \setminus A) = \pi_{-r}(\lambda) + \varepsilon(A),\end{aligned}$$

where $\varepsilon(A) \in \{0,1\}$ (note that for the last equation we have used again that $r \neq 0$ and p is odd, so that $r \neq -r$). Hence we conclude

$$\pi_r(\lambda) - \pi_{-r}(\lambda) - 1 = \varepsilon(B) + 1 = \varepsilon(A);$$

thus we deduce $\varepsilon(B) = 0$ and $\varepsilon(A) = 1$.

Now by removing a removable node of residue r, we can either create new removable nodes of residue $r \pm 1$ or indent nodes of such residues can vanish; the overall effect is in all cases to lift the paths for these residues by 1 at this position.

If A and B both are higher (resp., both are lower) than the good node of such residues, then the corresponding peak value in $\lambda \setminus A$ and $\lambda \setminus B$ is increased in both cases by 1 (resp., is in both cases unchanged) compared to the peak value in λ. Since B is normal of the same residue as the good node A, it is higher in λ than A; hence we also might have an increase in the critical peak value by 1 in $\lambda \setminus B$ and an unchanged peak value when removing A (but not the other way around!). Hence the situation $\varepsilon(B) = 0$ and $\varepsilon(A) = 1$ is impossible, and thus the claim is proved. ∎

The theorem above allows us to deduce information on the restrictions of the A_n-representations; as in characteristic 0, we denote by $\widehat{C^\lambda}$ the module $C^{\lambda^+} \oplus C^{\lambda^-}$ if $\lambda = \lambda^M$ (resp., the module $C^\lambda = C^{\lambda^M}$ if $\lambda \neq \lambda^M$).

THEOREM 5.4. *Let λ be p-regular with $\lambda \neq \lambda^M$, and assume that λ has a good node A with $\lambda \setminus A = (\lambda \setminus A)^M$. Then*

(i)

$$\operatorname{soc}\big(C^\lambda|_{A_{n-1}}\big) \simeq \widehat{C^{\lambda \setminus A}} \oplus \bigoplus_{\substack{B \neq A \\ \text{good}}} C^{\lambda \setminus B},$$

and this socle is multiplicity-free.

(ii) *Let B be a removable node of λ such that $\lambda \setminus B$ is p-regular. Then the multiplicity of $C^{\lambda \setminus B^o}$ in $C^\lambda|_{A_{n-1}}$ is given by*

$$\big[C^\lambda|_{A_{n-1}} : C^{\lambda \setminus B^o}\big] = \begin{cases} \operatorname{ht} A & \text{if } B = A, \\ \operatorname{ht} B + \big[D^\lambda|_{S_{n-1}} : D^{(\lambda \setminus B)^M}\big] & \text{else}. \end{cases}$$

Proof. This follows from the Branching Theorem for modular S_n-representations, the classification of the modular irreducible A_n-representations, and part (ii) of Theorem 5.3. ∎

Remark. Note that in the case of $B \neq A$ in part (ii) above, we do not know how to compute the second summand.

THEOREM 5.5 (The far-from-p-symmetric case). *Let λ be p-regular with $\lambda \neq \lambda^M$, and assume that λ has no good node A with $\lambda \setminus A = (\lambda \setminus A)^M$. Then there are no good nodes B and $\tilde{B}$ with*

$$\lambda \setminus B = \left(\lambda \setminus \tilde{B}\right)^M.$$

Proof. Assume there are good nodes B and $\tilde{B}$ with

$$(*) \qquad \lambda \setminus B = \left(\lambda \setminus \tilde{B}\right)^M.$$

Then these nodes are different from A, and their residues $r = \text{res } B$, $s = \text{res } \tilde{B}$ are different, and both are different from 0.

We consider again the p-content vector $c(\lambda) = (c_0, c_1, \ldots, c_{p-1})$ of λ, and then compare the content vectors of $\lambda \setminus B$ and $(\lambda \setminus \tilde{B})^M$. For $s \neq p - r$, this gives the equations

$$c_{p-r} = c_r - 1, c_r = c - p - r$$

and hence a contradiction. Thus we must have $s = p - r$, i.e., B and $\tilde{B}$ are good nodes with conjugate residues.

Next we consider the residue symbols of $\lambda \setminus B$ and $(\lambda \setminus \tilde{B})^M$. Let the residue symbol of λ be

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{Bmatrix}.$$

Since B and $\tilde{B}$ are good (and not of residue 0), their removal from λ only has an effect on an entry r (resp., $p - r$) in the residue symbol at the ith and jth columns (say).

Let us first consider the case where the ith column in λ is

$$\begin{matrix} \boxed{r} \\ y \end{matrix},$$

with the marked entry corresponding to the good node B. If $i \neq j$, then $(*)$ implies that the ith column in $R_p(\lambda \setminus B)$ satisfies

$$\begin{matrix} r-1 \\ y \end{matrix} = \begin{cases} \begin{matrix} -y \\ -r \end{matrix} & \text{if } y \neq 1+r, \\ \text{or} & \\ \begin{matrix} -r \\ 1-r \end{matrix} & \text{if } y = 1+r. \end{cases}$$

In both cases we immediately obtain a contradiction. If $i = j$, then the ith column in λ is

$$\begin{matrix} \boxed{r} \\ \boxed{-r} \end{matrix},$$

with the second marked entry corresponding to the good node $\tilde{B}$; hence this column is Mullineux-fixed. But by $(*)$, all other columns in the residue symbol are also Mullineux-fixed, and hence so is λ, a contradiction.

Next we first consider the case where the ith column in λ is

$$\begin{matrix} x \\ \boxed{r} \end{matrix},$$

with the marked entry corresponding again to the good node B; note that this column then has to be regular. If $i \neq j$, then $(*)$ implies now that the ith column in $R_p(\lambda \setminus B)$ satisfies

$$\begin{matrix} x \\ r+1 \end{matrix} = \begin{matrix} -r \\ -x \end{matrix},$$

immediately giving a contradiction. If $i = j$, then the ith column in λ is now

$$\begin{matrix} \boxed{-r} \\ \boxed{r} \end{matrix};$$

hence again this column is Mullineux-fixed. But as before, $(*)$ implies that all other columns in the residue symbol are also Mullineux-fixed; hence so is λ, again a contradiction. ∎

From this we deduce the following.

THEOREM 5.6. *Let λ be p-regular with $\lambda \neq \lambda^M$, and assume that λ has no good node A with $\lambda \setminus A = (\lambda \setminus A)^M$. Then*

(i)

$$\operatorname{soc}\left(C^{\lambda}|_{A_{n-1}}\right) \simeq \bigoplus_{\substack{B \\ \text{good}}} C^{\lambda \setminus B},$$

and this socle is multiplicity-free.

(ii) *Let B be a removable node of λ such that $\lambda \setminus B$ is p-regular. Then the multiplicity of $C^{\lambda \setminus B}$ in $C^{\lambda}|_{A_{n-1}}$ is given by*

$$\left[C^{\lambda}|_{A_{n-1}} : C^{\lambda \setminus B}\right] = \operatorname{ht} B + \left[D^{\lambda}|_{S_{n-1}} : D^{(\lambda \setminus B)^M}\right].$$

THEOREM 5.7 (The p-symmetric case). *Let λ be p-regular with $\lambda = \lambda^M$.*

(i) *If N is a removable node of λ with $\lambda \setminus N = (\lambda \setminus N)^M$, then* res $N = 0$.

(ii) *For any good node B there is a good node $\tilde{B}$ with* res $B = -$res $\tilde{B}$, *and*

$$\lambda \setminus B = \left(\lambda \setminus \tilde{B}\right)^M.$$

In particular, a good node A in λ of residue 0 *satisfies $\lambda \setminus A = (\lambda \setminus A)^M$.*

(iii) *If λ has only one good node, then this is of residue* 0.

Proof. (i) We consider the content vector $c(\lambda) = (c_0, \ldots, c_{p-1})$. As $\lambda = \lambda^M$, we have $c_i = c_{p-i}$ for all i. Let N be a removable node of residue $r \neq 0$; if $\lambda \setminus N = (\lambda \setminus N)^M$, then this implies $c_r - 1 = c_{p-r}$, and hence a contradiction.

(ii) If B is a good node of λ of residue r, then we have

$$(\lambda \setminus B)^M = \lambda^M \setminus B' = \lambda \setminus B',$$

where B' is a good node of λ of conjugate residue $p - r$. If B is good of residue $r = 0$, then we must have $B' = B$.

(iii) follows immediately from (ii). ∎

For the branching of the representations this implies

THEOREM 5.8. *Let λ be p-regular with $\lambda = \lambda^M$. Then we have*

(i) *The restrictions of the two conjugate modules $C^{\lambda+}$ and $C^{\lambda-}$ have conjugate socles*

$$\left(C^{\lambda \setminus A_0}\right)^{\pm} \oplus \bigoplus_{\substack{B \text{ good} \\ \operatorname{res} B \in \{1, \ldots, (p-1)/2\}}} C^{\lambda \setminus B},$$

where the first summand only occurs if λ *has a good node* A_0 *of residue* 0. *In particular, these restrictions are multiplicity-free.*

(ii) *Let* B *be a normal node of* λ *such that* $\lambda \setminus B$ *is* p*-regular. Then*

$$\left[\widehat{C^{\lambda}}\big|_{A_{n-1}} : C^{\lambda \setminus B^{o}}\right] = \begin{cases} 2\,\mathrm{ht}\, B & \text{if } (\lambda \setminus B)^{M} \neq \lambda \setminus B, \\ \mathrm{ht}\, B & \text{if } (\lambda \setminus B)^{M} = \lambda \setminus B. \end{cases}$$

As has been described in Section 3, we know by Kleshchev's Branching Theorems which irreducible S_n-modules split on restriction to S_{n-1}. We are now going to use this together with the description of the residue symbols of JS-partitions given in [3] for dealing with the analogue of the Jantzen–Seitz question for the alternating groups. We first recall the description of JS-partitions that are fixed under the Mullineux map.

THEOREM 5.9. *Let* λ *be a* p*-regular partition. Then* λ *is a JS-partition and a Mullineux fixed point if and only if its residue symbol,*

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{Bmatrix},$$

can be constructed iteratively by the following procedure:

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

If the first $k-1$ *columns of the residue symbol are already constructed, then we have two possibilities for a regular extension, namely*

$$\begin{Bmatrix} x_k \\ y_k \end{Bmatrix} = \begin{Bmatrix} 1 - y_{k-1} \\ y_{k-1} - 1 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} y_{k-1} - 1 \\ 1 - y_{k-1} \end{Bmatrix},$$

and if $y_{k-1} = 1$, *then we have, furthermore, one possibility for a singular extension, namely,*

$$\begin{Bmatrix} x_k \\ y_k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

The construction rule given in the theorem can be described alternatively by a diagram in the following way. The JS-partitions that are Mullineux fixed points are obtained by adding on columns to the residue symbol according to a walk starting at

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

in the diagram $(*)$ below (we omit the brackets for simplification); the symbol $\subset$ in the diagram means that we have a loop at the corresponding node of the graph.

$$(*) \qquad \subset \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{array}{ccccccccccc} \nearrow & \begin{matrix} 0 \\ 0 \end{matrix} & \rightarrow & \begin{matrix} 1 \\ -1 \end{matrix} & \rightarrow & \begin{matrix} 2 \\ -2 \end{matrix} & \rightarrow & \cdots & \rightarrow & \begin{matrix} \frac{p-3}{2} \\ -\frac{p-3}{2} \end{matrix} \\ & \updownarrow & & \updownarrow & & \updownarrow & & \cdots & & \updownarrow \\ \nwarrow & \begin{matrix} -1 \\ 1 \end{matrix} & \leftarrow & \begin{matrix} -2 \\ 2 \end{matrix} & \leftarrow & \begin{matrix} -3 \\ 3 \end{matrix} & \leftarrow & \cdots & \leftarrow & \begin{matrix} -\frac{p-1}{2} \\ \frac{p-1}{2} \end{matrix} \end{array}$$

Since the starting column

$$\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$$

already guarantees a normal node of residue 0, it is clear that the unique good node of such JS partitions is of residue 0, i.e., that they are of type 0. Furthermore, Mullineux fixed JS-partitions have empty or square p-cores, and the p-core only depends on the end of the walk in the diagram above (see [3]). Thus we can give the p-cores of Mullineux fixed JS-partitions by putting the corresponding p-core at the end point of the walk in the diagram:

$$\subset \emptyset \quad \begin{array}{ccccccccccc} \nearrow & (1) & \rightarrow & (2^2) & \rightarrow & (3^3) & \rightarrow & \cdots & \rightarrow & (\frac{p-1}{2}^{\frac{p-1}{2}}) \\ & \updownarrow & & \updownarrow & & \updownarrow & & \cdots & & \updownarrow \\ \nwarrow & \emptyset & \leftarrow & (1) & \leftarrow & (2^2) & \leftarrow & \cdots & \leftarrow & (\frac{p-3}{2}^{\frac{p-3}{2}}) \end{array}$$

THEOREM 5.10. *Let $n \geq 2$, let p be an odd prime, and let λ be a p-regular partition of n. Then the following are equivalent*:

(i) $C^{\lambda^o}|_{A_{n-1}}$ *is irreducible.*

(ii) *One of the following holds*:

(a) $D^{\lambda}|_{S_{n-1}}$ *is irreducible.*

(b) $D^{\lambda}|_{S_{n-1}} \simeq D^{\lambda(1)} \oplus D^{\lambda(1)^M}$ *and* $\lambda^M = \lambda$ *but* $\lambda(1)^M \neq \lambda(1)$.

(iii) *One of the following holds*:

(a) λ *is a JS-partition.*

(b) *The residue symbol $R_p(\lambda)$ is obtained by adding on columns along a walk in the extended diagram $(**)$ below, starting at any regular column* ${}^{\;x}_{-x}$, *with $x \neq 0$.*

(iii′) *One of the following holds*:

(a) λ *is a JS-partition that is not Mullineux fixed.*

(b) $R_p(\lambda)$ *is obtained by adding on columns along a walk in the diagram* $(*)$ *above starting at* $\genfrac{}{}{0pt}{}{0}{0}$, *or along a walk in the extended diagram* $(**)$ *below, starting at any regular column* $\genfrac{}{}{0pt}{}{x}{-x}$, *with* $x \neq 0$.

$(**)$

$$
\begin{array}{cccccccccccccc}
 & & \genfrac{}{}{0pt}{}{0}{0} & \rightarrow & & \genfrac{}{}{0pt}{}{1}{-1} & \rightarrow & & \genfrac{}{}{0pt}{}{2}{-2} & \rightarrow & \cdots & \rightarrow & & \genfrac{}{}{0pt}{}{\frac{p-3}{2}}{-\frac{p-3}{2}} \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & & & \searrow & \nearrow & \\
\subset \genfrac{}{}{0pt}{}{0}{1} & & \updownarrow & & \subset \genfrac{}{}{0pt}{}{0}{1} & & \updownarrow & \subset \genfrac{}{}{0pt}{}{0}{1} & & \updownarrow & \cdots & & \subset \genfrac{}{}{0pt}{}{0}{1} & \updownarrow \\
 & \nwarrow & & \swarrow & \nwarrow & & \swarrow & \nwarrow & & & & \swarrow & \nwarrow & \\
 & & \genfrac{}{}{0pt}{}{-1}{1} & \leftarrow & & \genfrac{}{}{0pt}{}{-2}{2} & \leftarrow & & \genfrac{}{}{0pt}{}{-3}{3} & \leftarrow & \cdots & \leftarrow & & \genfrac{}{}{0pt}{}{-\frac{p-1}{2}}{\frac{p-1}{2}}
\end{array}
$$

Proof. First we assume (i). If $\lambda^M \neq \lambda$, then $D^\lambda|_{A_{n-1}} \simeq C^\lambda|_{A_{n-1}}$ is irreducible, so this implies immediately that $D^\lambda|_{S_{n-1}}$ is irreducible.

So we now consider a fixed point $\lambda^M = \lambda$. Note that C^{λ^+} and C^{λ^-} are conjugate representations, so by assumption the conjugate modules $C^{\lambda^+}|_{A_{n-1}}$ and $C^{\lambda^-}|_{A_{n-1}}$ are also both irreducible.

If $D^\lambda|_{S_{n-1}}$ is irreducible, there is nothing to prove.

In the other case we deduce by Proposition 5.2 and Kleshchev's Branching Theorem:

$$D^\lambda|_{S_{n-1}} \simeq D^{\lambda \setminus A} \oplus D^{\lambda \setminus B},$$

where A and B are the two good nodes of λ. Moreover, since $\lambda = \lambda^M$, we must have $(\lambda \setminus A)^M = \lambda \setminus B$. Since we always have a normal node at the first corner of a partition, giving a composition factor of the restriction in the block corresponding to the removal of the good node of the same residue, the first removable node itself must be good. Hence we obtain

$$D^\lambda|_{S_{n-1}} \simeq D^{\lambda(1)} \oplus D^{\lambda(1)^M},$$

with $\lambda(1) \neq \lambda(1)^M$.

Thus (i) $\Rightarrow$ (ii) is proved.

For (ii) $\Rightarrow$ (iii) we have to show that a non-JS-partition λ with $\lambda = \lambda^M$, $\lambda(1) \neq \lambda(1)^M$, and

$$D^\lambda|_{S_{n-1}} \simeq D^{\lambda(1)} \oplus D^{\lambda(1)^M}$$

can be constructed along the diagram $(**)$ above. Note that $\lambda(1)^M = \lambda(g)$ for some $g \neq 1$, corresponding to removing a good node at the gth corner of λ.

Consider the residue symbol

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{Bmatrix}.$$

The condition $\lambda = \lambda^M$ amounts to the equations

$$\begin{array}{ll} x_i = -y_i & \text{for regular columns (note that } x_i \neq (p-1)/2\text{).} \\ x_i = 0, y_i = 1 & \text{for singular columns (here we must have } i \neq 1\text{).} \end{array}$$

We know from the decomposition of $D^\lambda|_{S_{n-1}}$ that λ has only the good nodes $\lambda \setminus \lambda(1)$ and $\lambda \setminus \lambda(g)$, and no further normal nodes [11]. Note also that the two good nodes must have conjugate residues $r \neq -r$, say.

Now consider the Mullineux sequence for λ:

$$\begin{array}{llllll} M(\lambda) = 0- & x_1 + & (x_1+1)- & (-x_1)+ & (-(x_1+1))- \\ & x_2 + & (x_2+1)- & y_2 + & (y_2+1)- \\ & \vdots & & & & . \\ & x_k + & (x_k+1)- & y_k + & (y_k+1)- \end{array}$$

If $x_1 = 0$, then $y_1 + = (-x_1)+$ gives a normal node of residue 0 (hence *the* good one of residue 0), contradicting the above properties of our two good nodes.

Hence $x_1 \neq 0$ is the first good node. Since $x_1 \neq (p-1)/2$, we have $-x_1 \neq x_1 + 1$, and hence $-x_1 = y_1$ is the next good node. Since we then have no further normal node, we must have $x_2 = x_1 + 1$ or $-(x_1+1)$ or 0.

If $x_2 = 0$, then $y_2 = 0$ is not possible, but only $y_2 = 1$, i.e., a singular column, and then the situation for the next column in the residue symbol is as before. Thus we have a certain number of singular columns $\{{0 \atop 1}\}$, and then the next regular column $\{{x_{i_2} \atop y_{i_2}}\}$ has to satisfy $x_{i_2} = x_1 + 1$ or $-(x_1+1)$, $y_{i_2} = -x_{i_2}$. Then again, we may have a number of singular columns, and then the next regular column $\{{x_{i_3} \atop y_{i_3}}\}$ satisfies $x_{i_3} = x_{i_2} + 1$ or $-(x_{i_2}+1)$, $y_{i_3} = -x_{i_3}$. This repeats, so we obtain indeed a walk in the extended diagram $(**)$, starting at a regular column different from ${0 \atop 0}$.

Now to (iii) $\Rightarrow$ (i).

Case 1. Assume λ is a JS-partition with $\lambda \neq \lambda^M$. If $\lambda(1) \neq \lambda(1)^M$, then clearly,

$$D^\lambda|_{A_{n-1}} \simeq D^{\lambda(1)}|_{A_{n-1}} \simeq C^\lambda|_{A_{n-1}}$$

is irreducible. So we may assume now that $\lambda(1) = \lambda(1)^M$.

Consider the residue symbol

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{Bmatrix}.$$

By Theorem 3.6 we know that the residue symbol $R_p(\lambda(1))$ of $\lambda(1)$ can only be one of

$$\begin{Bmatrix} x_1 - 1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x_1 & \cdots & x_k \\ y_1 + 1 & \cdots & y_k \end{Bmatrix}$$

$$\text{or} \quad \begin{Bmatrix} x_2 & \cdots & x_k \\ y_2 & \cdots & y_k \end{Bmatrix},$$

where

$$\begin{Bmatrix} x_1 - 1 \\ y_1 \end{Bmatrix}^M = \begin{Bmatrix} x_1 - 1 \\ y_1 \end{Bmatrix} \quad \text{resp.,} \begin{Bmatrix} x_1 \\ y_1 + 1 \end{Bmatrix}^M = \begin{Bmatrix} x_1 \\ y_1 + 1 \end{Bmatrix}$$

$$\text{resp., } x_1 = y_1 = 0,$$

and furthermore,

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}^M \neq \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix},$$

and all other columns are Mullineux-fixed and hence of the form

$$\begin{Bmatrix} x \\ -x \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

By these conditions, the second and third possibilities for the residue symbol of $\lambda(1)$ are immediately excluded; in particular, this implies that the type α of λ satisfies $\alpha \neq 0$. So we have the following possibilities now for the first column of $R_p(\lambda)$:

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \in \left\{ \begin{Bmatrix} \alpha \\ \alpha + 1 \end{Bmatrix}, \alpha \neq 0; \begin{Bmatrix} \alpha \\ 0 \end{Bmatrix}, \alpha \neq 0, p - 1; \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix}, \alpha \neq 0, 1 \right\}.$$

Then $R_p(\lambda(1))$ starts with one of

$$\begin{Bmatrix} \alpha - 1 \\ \alpha + 1 \end{Bmatrix}, \qquad \begin{Bmatrix} \alpha - 1 \\ 0 \end{Bmatrix}, \qquad \begin{Bmatrix} 0 \\ \alpha + 1 \end{Bmatrix},$$

respectively; but the first one is not Mullineux-fixed, the second is only for $\alpha = 1$, and the third is only for $\alpha = p - 1$; hence $R_p(\lambda)$ has to start with $\left\{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\}$ and $\alpha = 1$, or with $\left\{\begin{smallmatrix} 0 \\ p-1 \end{smallmatrix}\right\}$ and $\alpha = p - 1$.

The restrictions for the further columns do not allow any regular extensions from these first columns. But also, the construction rules for singular columns do not give Mullineux fixed columns as required. Hence we must have

$$R_p(\lambda) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} 0 \\ p-1 \end{Bmatrix},$$

which corresponds to $\lambda = (2)$ (resp., $\lambda = (1^2)$), and then (i) obviously holds.

Case 2. We assume now that λ is a JS-partition with $\lambda = \lambda^M$. Then λ is of type 0, and it is immediately seen that also $\lambda(1) = \lambda(1)^M$, and then (i) holds.

Case 3. Assume now that λ is not a JS-partition, so λ is constructed along a path in diagram $(**)$, in particular, $\lambda = \lambda^M$. Reversing the arguments for (ii) $\Rightarrow$ (iii), we have

$$D^\lambda|_{S_{n-1}} \simeq D^{\lambda \setminus A} \oplus D^{\lambda \setminus B},$$

where A, B are good nodes of λ of conjugate residues $\neq 0$, and $(\lambda \setminus A)^M = \lambda \setminus B$. Then $C^{\lambda^o}|_{A_{n-1}} \simeq D^{\lambda \setminus N}|_{A_{n-1}}$, $N \in \{A, B\}$, and hence this restriction is irreducible.

The equivalence of (iii) and (iii′) is clear. ∎

From the proof of the theorem, one easily deduces the following.

COROLLARY 5.11. *We assume the notation of the theorem. Then if $C^{\lambda^o}|_{A_{n-1}}$ is irreducible, we have $C^{\lambda^o}|_{A_{n-1}} \simeq C^{\lambda(1)^o}$.*

Remarks. (i) In part (iii) of the theorem above, it would be nice to have a description in terms of the parts of λ like the one available for JS-partitions.

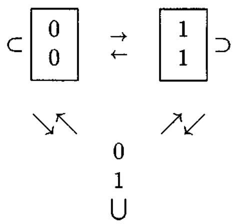
(ii) We have already observed that JS-partitions have special p-cores [3]; moreover, the "almost JS-partitions" constructed along a path in $(**)$ (resp., also those constructed in a more general fashion in the construction diagram for JS-partitions [3]) have special cores that depend in an easy way on the start and the end of the path in the diagram.

6. BRANCHING AT $p = 2$

As we have already seen in Section 4, the behavior of the A_n-representations at characteristic 2 is very much different from that at odd primes. While in some respects the combinatorics of S-partitions is easier than that of Mullineux fixed points, the 2-modular representation theory presents more difficulties, since A_n is a normal subgroup of index $p = 2$ in S_n.

First we give a description of S-partitions (see Theorem 4.1) in terms of residue symbols as an analogue of the description of Mullineux fixed points via residue symbols.

PROPOSITION 6.1. *The* 2-*residue symbols of S-partitions are constructed by a walk in the following directed graph, where we start at one of the marked columns and we may loop around any of the three vertices in the graph*:



In other words, the residue symbols start with a regular column, and they have no column $\genfrac{}{}{0pt}{}{1}{0}$.

Proof. As long as the smallest part of λ is at least 2, the 2-rim consists of horizontal dominoes and thus is 2-singular. In removing such singular 2-rims from a given S-partition λ, we again obtain S-partitions. Doing this as long as possible, we reach an S-partition, which we may assume to be of even length by adding a part 0 if necessary. This S-partition then ends on one of the following pairs of parts:

$$1,0 \quad 2,1 \quad 3,1.$$

Then the next step in removing the 2-rim leads to the following corresponding columns in the residue symbol:

$$\begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix},$$

and the partition obtained after the removal is again an S-partition.

So S-partitions can be built up along residue symbols. As we can never start a residue symbol with the singular column $\genfrac{}{}{0pt}{}{0}{1}$, and since $\genfrac{}{}{0pt}{}{1}{0}$ is not an S-partition, the start has to be at one of the two marked columns.

Using the list of end pairs given above, it is then easily seen that at each step all three columns in the diagram above give possible extensions to an S-partition. ∎

COROLLARY 6.2. *Let λ be an S-partition with residue symbol $R_2(\lambda)$. Let m_0 be the number of columns $\genfrac{}{}{0pt}{}{0}{0}$ in $R_2(\lambda)$ (resp., let m_1 be the number of columns $\genfrac{}{}{0pt}{}{1}{1}$). Then the 2-core of λ is given by*

$$\lambda_{(2)} = \begin{cases} (2(m_0 - m_1) - 1, 2(m_0 - m_1) - 2, \ldots, 2, 1) & \textit{if } m_0 - m_1 > 0, \\ (2(m_1 - m_0), 2(m_1 - m_0) - 1, \ldots, 2, 1) & \textit{if } m_1 - m_0 > 0, \\ \varnothing & \textit{if } m_0 = m_1. \end{cases}$$

Proof. Note that the p-content $c = (c_0, c_1, \ldots, c_{p-1})$ of a partition determines its p-core. The associated $\vec{n}$-vector is given by $\vec{n} = (c_0 - c_1, c_1 - c_2, \ldots, c_{p-2} - c_{p-1}, c_{p-1} - c_0)$. By [2] we can easily compute the $\vec{n}$-vector of λ via its residue symbol. Hence we obtain the 2-core of the partition λ given via its residue symbol $R_2(\lambda)$ as above; we omit the details. ∎

LEMMA 6.3. *Let λ be a 2-regular partition.*

(a) *If λ is an S-partition and A is a removable node of λ such that $\lambda \setminus A$ is 2-regular, then we have*

$$\lambda \setminus A \textit{ is an S-partition if and only if } \operatorname{res} A = 0.$$

(b) *If λ is not an S-partition, then there is at most one node A in λ such that $\lambda \setminus A$ is an S-partition, and such a node is of residue* 1.

Proof. By definition, the parts of S-partitions come in one of the following pairs (pictured in the 2-residue diagram):

	⋯		0
	⋯	0	

	⋯		1
	⋯	1	

	⋯			0
	⋯	1		

From this, (a) and (b) easily follow. ∎

Analogous to the notation at odd characteristic, we let $C^{\lambda^o} = C^\lambda$ if λ is not an S-partition and $C^{\lambda^o} \in \{C^{\lambda^+}, C^{\lambda^-}\}$ if λ is an S-partition. For the A_n-representations at characteristic 2, we can now deduce the following:

THEOREM 6.4. *Let λ be a 2-regular partition of n.*

(a) *Assume λ is an S-partition of n. Then the module* $\operatorname{soc}(C^{\lambda^\pm}|_{A_{n-1}})$ *has a constituent $C^{(\lambda \setminus A_0)^\pm}$ if A_0 is a good node of residue* 0, *and it has a constituent $C^{\lambda \setminus A_1}$ if A_1 is a good node of residue* 1.

Furthermore, if B is a removable node with $\lambda \setminus B$ 2-regular, then

$$\left[\widehat{C^\lambda}|_{A_{n-1}} : C^{\lambda \setminus B^o}\right] = \begin{cases} \operatorname{ht} B & \textit{if } B \textit{ is normal}, \\ 0 & \textit{else}, \end{cases}$$

and $\operatorname{ht} B$ *is even for all normal nodes B of residue* 1 *such that $\lambda \setminus B$ is 2-regular.*

(b) *Assume* λ *is not an S-partition. Then the module* $\operatorname{soc}(C^{\lambda}|_{A_{n-1}})$ *has a constituent* $C^{\lambda\setminus A_0}$ *if* A_0 *is a good node of residue* 0, *and it has a constituent* $C^{\lambda\setminus A_1}$ *if* A_1 *is a good node of residue* 1 *with* $\lambda\setminus A_1$ *not an S-partition* (*resp., a constituent* $C^{\widehat{\lambda\setminus A_1}}$ *if* A_1 *is a good node of residue* 1 *with* $\lambda\setminus A_1$ *an S-partition*).

The latter occurs only if λ *is almost an S-partition except for one pair of parts* $(2k, 2k-2)$, *with* A_1 *at the end of the first of these two parts.*

Furthermore, if B is a removable node with $\lambda\setminus B$ 2-*regular, then*

$$\left[C^{\lambda}|_{A_{n-1}} : C^{\lambda\setminus B^{o}}\right] = \begin{cases} \operatorname{ht} B & \text{if } B \text{ is normal,} \\ 0 & \text{else.} \end{cases}$$

The case $C^{\lambda\setminus B^{o}} = C^{\lambda\setminus B^{\pm}}$ *arises only if* λ *is almost an S-partition except for one pair of parts of the form* $(2k, 2k-2)$ *or* $(2k, 2k-3)$, *with B at the end of the part* $2k$ *normal of residue* 1.

For $p = 2$, the characterization of the irreducible restrictions is not as complete as for odd primes p, but there are at least very strong combinatorial restrictions. First we compare the A_n-situation with the S_n-situation:

THEOREM 6.5. *Let* λ *be a* 2-*regular partition. Then the following are equivalent*:

(i) $C^{\lambda^{o}}|_{A_{n-1}}$ *is irreducible.*

(ii) *One of the following holds*:

(a) $D^{\lambda}|_{S_{n-1}}$ *is irreducible and* $\lambda \neq (2l, 2l-2)$ *if* $n \equiv 2 \pmod 4$.

(b) $D^{\lambda}|_{S_{n-1}} \simeq \begin{smallmatrix} D^{\lambda(2)} \\ D^{\lambda(2)} \end{smallmatrix}$ (*where this denotes a uniserial module with top and socle* $D^{\lambda(2)}$), *and* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *is an S-partition with* λ_1 *even.*

Proof. First we assume (i).

Case 1. Suppose λ is not an S-partition; hence $D^{\lambda}|_{A_{n-1}} \simeq C^{\lambda}|_{A_{n-1}}$ is irreducible, and so $D^{\lambda}|_{S_{n-1}} \simeq D^{\lambda(1)}$ must be irreducible, i.e., λ is a JS-partition. Furthermore, as $D^{\lambda(1)}|_{A_{n-1}}$ is irreducible, also $\lambda(1)$ is not an S-partition. Thus λ is not of the form $(2l, 2l-2)$ for some $l > 1$, as was to be proved in this case.

Case 2. Suppose λ is an S-partition; hence

$$D^{\lambda}|_{A_n} \simeq C^{\lambda^{+}} \oplus C^{\lambda^{-}},$$

with $C^{\lambda^{+}} \not\simeq C^{\lambda^{-}}$. Now

$$D^{\lambda}|_{A_{n-1}} \simeq C^{\lambda^{+}}|_{A_{n-1}} \oplus C^{\lambda^{-}}|_{A_{n-1}},$$

a direct sum of two conjugate irreducibles; hence we have one of the following two possibilities:

(a) λ is a JS-partition and $\lambda(1)$ is an S-partition.

(b) $D^{\lambda}|_{S_{n-1}} \sim 2 \cdot D^{\mu}$, where μ is not an S-partition.

Case (a). As $\lambda = (\lambda_1, \dots, \lambda_k)$ is a JS-partition, we have $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_k \pmod 2$. Since λ is also an S-partition, we have

$$\lambda_{2j-1} - \lambda_{2j} = 2 \qquad \text{and} \qquad \lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_k \equiv 1 \pmod 2.$$

But then $\lambda(1)$ is always an S-partition, so this condition can be omitted in (a). Note also that in this case again, $\lambda = (2l, 2l-2)$ does not occur.

Case (b). Since the socle of $D^{\lambda}|_{S_{n-1}}$ is multiplicity-free, the module $D^{\lambda}|_{S_{n-1}}$ cannot be completely reducible. Hence we must have

$$\mathrm{soc}\left(D^{\lambda}|_{S_{n-1}}\right) \simeq D^{\lambda(i)},$$

where $\mu = \lambda(i) = \lambda \setminus A$, A being the only good node of λ. Furthermore, the Branching Theorem tells us that there is no normal node of residue $\neq$ res A, and there is exactly one other normal node B of residue res A; moreover, for this normal node B, the partition $\lambda \setminus B$ is not p-regular, and we have

$$D^{\lambda}|_{S_{n-1}} \simeq \begin{matrix} D^{\lambda(i)} \\ D^{\lambda(i)} \end{matrix}.$$

Since the first removable node of λ is always normal, this must be the normal node B. Note also that $\lambda(i)$ is not an S-partition, since otherwise we have a contradiction to condition (i) of the theorem.

Hence we have $\lambda_1 - \lambda_2 = 1$, so $\mathrm{res}(\lambda \setminus \lambda(1)) = \mathrm{res}(\lambda \setminus \lambda(2))$, and thus $\lambda \setminus \lambda(2)$ is also normal, and hence it has to be the good node A. Thus we must have $i = 2$, i.e.,

$$D^{\lambda}|_{S_{n-1}} \simeq \begin{matrix} D^{\lambda(2)} \\ D^{\lambda(2)} \end{matrix},$$

as claimed. Furthermore, as $\lambda(2)$ is not an S-partition, the S-partition λ must start with an even part.

Now assume that (ii) is satisfied.

Case 1. First suppose that $D^{\lambda}|_{S_{n-1}}$ is irreducible, so λ is a JS-partition, but λ is not of the form $(2l, 2l-2)$ for some $l > 1$. If λ is an S-partition, then $\lambda(1)$ is an S-partition by the same argument as in Case (a) above.

Hence

$$\left(D^{\lambda}|_{S_{n-1}}\right)_{A_{n-1}} \simeq D^{\lambda}|_{A_{n-1}} \simeq C^{\lambda^{+}}|_{A_{n-1}} \oplus C^{\lambda^{-}}|_{A_{n-1}}$$
$$\simeq D^{\lambda(1)}|_{A_{n-1}} \simeq C^{\lambda(1)^{+}} \oplus C^{\lambda(1)^{-}},$$

and so both $C^{\lambda^{+}}|_{A_{n-1}}$ and $C^{\lambda^{-}}|_{A_{n-1}}$ are irreducible.

If $\lambda = (\lambda_1, \dots, \lambda_k)$ is not an S-partition, then $\lambda(1)$ is also not an S-partition. To see this, assume that $\lambda(1) = (\lambda_1 - 1, \dots, \lambda_k)$ is an S-partition. As λ is a JS-partition, we have $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_k \pmod 2$, and hence $\lambda_1 + \lambda_2 \equiv 2 \pmod 4$, $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod 4$ for $j \neq 1$, and $\lambda_{2j-1} - \lambda_{2j} = 2$ for all j.

Hence we must have $\lambda_j \equiv 0 \pmod 2$ for all j, but then $\lambda_{2j-1} + \lambda_{2j} \equiv 2 \pmod 4$ for all j, and this implies that λ has only two parts, i.e., $\lambda = (2l, 2l - 2)$. But this case was excluded in condition (ii) of the theorem.

Having shown all these properties of λ and $\lambda(1)$, it is now clear that condition (i) of the theorem is satisfied.

Case 2. We now consider the situation where λ is an S-partition with even first part λ_1 and

$$D^{\lambda}|_{S_{n-1}} \simeq \begin{array}{c} D^{\lambda(2)} \\ D^{\lambda(2)} \end{array}.$$

Here $\lambda = (\lambda_1, \dots, \lambda_k)$ must satisfy $\lambda_1 - \lambda_2 = 1$, λ_1 even, and then $\lambda(2) = (\lambda_1, \lambda_1 - 2, \dots)$ is not an S-partition. Hence

$$D^{\lambda}|_{A_{n-1}} \simeq \left. \begin{array}{c} D^{\lambda(2)} \\ D^{\lambda(2)} \end{array} \right|_{A_{n-1}} \sim 2 \cdot C^{\lambda(2)},$$

and thus

$$C^{\lambda^{+}}|_{A_{n-1}} \simeq C^{\lambda^{-}}|_{A_{n-1}} \simeq C^{\lambda(2)}$$

is irreducible, as was to be proved. ∎

Remark. Note that the proof of the theorem tells us exactly what the restriction $C^{\lambda^{o}}|_{A_{n-1}}$ is in the cases when it is irreducible.

For a partition $\mu = (\mu_1, \dots, \mu_m)$ of n into distinct parts, we define its *doubling* to be the partition

$$\operatorname{dbl}(\mu) = \left(\left\lceil \frac{\mu_1 + 1}{2} \right\rceil, \left\lfloor \frac{\mu_1}{2} \right\rfloor, \left\lceil \frac{\mu_2 + 1}{2} \right\rceil, \left\lfloor \frac{\mu_2}{2} \right\rfloor, \dots, \left\lceil \frac{\mu_m + 1}{2} \right\rceil, \left\lfloor \frac{\mu_m}{2} \right\rfloor \right).$$

Via the process of regularization (see [6]), we then obtain a 2-regular partition $\operatorname{dbl}^2(\mu) := \operatorname{dbl}(\mu)^R$.

PROPOSITION 6.6. *Let $p = 2$, and let λ be a 2-regular partition satisfying one (and hence both) of the conditions in the theorem above. Then λ is a JS-partition $\neq (2l, 2l-2)$, or it is of the form $\lambda = \mathrm{dbl}^2(\mu)$, where μ is a partition of n into distinct parts satisfying $\mu_i - \mu_{i+1} > 4$ for $i = 1, \ldots, m-1$, and the residues of $\mu_1, \mu_2, \ldots$ modulo 4 follow a path in the diagram below, starting at* 3:

$$\begin{array}{ccccc} & & \overset{\cap}{0} & & \\ & \nearrow & & \searrow & \\ \boxed{3} & & \leftrightarrow & & 1 \end{array}$$

Proof. If λ is not a JS-partition, we have already seen in the proof of the theorem that λ has the following properties:

(i) λ is an S-partition.

(ii) λ_1 even.

(iii) $\lambda_1 - \lambda_2 = 1$.

(iv) The first two removable nodes are the only normal nodes of λ.

Property (i) immediately implies that $\lambda = \mathrm{dbl}^2(\mu)$ for some 2-regular partition $\mu = (\mu_1, \mu_2, \ldots)$ with $\mu_i - \mu_{i+1} \geq 4$ and $\mu_i - \mu_{i+1} > 4$ if $\mu_i \equiv 0 \pmod 4$, and $\mu_i \not\equiv 2 \pmod 4$ for all i. By properties (ii) and (iii), we have $\mu_1 \equiv 3 \pmod 4$.

Consider the sequence $r_1, r_2, \ldots$ of end residues of the rows of λ in its 2-residue diagram. By properties (ii) and (iv) we have

$$0 \leq \left|\{j \leq i | r_j = 1\}\right| - \left|\{j \leq i | r_j = 0\}\right| \leq 2.$$

Since a part $\mu_i \equiv 0$, 1 or 3 (mod 4) leads to the residue pairs $(0,1)$, $(0,0)$ resp., $(1,1)$, the condition above easily translates into the admissible walks in the triangle graph above, and it is clear that then $\mu_i - \mu_{i+1} > 4$ must hold for all i. ∎

Remarks. (a) There are examples of the phenomenon occurring in part (ii)(b) of Theorem 6.5 above. For all l we have

$$D^{(2l, 2l-1)}|_{S_{2l-2}} \simeq \frac{D^{(2l, 2l-2)}}{D^{(2l, 2l-2)}}.$$

This follows by using, e.g., [1].

There are also such examples with partitions with more than two parts:

$$D^{(6,5,1)}|_{S_{11}} \simeq \frac{D^{(6,4,1)}}{D^{(6,4,1)}} \qquad D^{(6,5,3,1)}|_{S_{14}} \simeq \frac{D^{(6,4,3,1)}}{D^{(6,4,3,1)}}.$$

(b) It is not clear whether the combinatorial condition in the proposition characterizes the partitions giving irreducible A_n-restrictions, thus providing an analogue of (iii) in Theorem 5.10.

(c) By the description of the partitions given in the proposition above, it is clear that we can have the irreducible restrictions of modular A_n-representations of the second type only if $n \equiv 0$ or $3 \pmod 4$, and the corresponding irreducible representations belong to the principal 2-block resp., to the 2-block with 2-core $(2, 1)$.

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